Selected works of Giuseppe Peano

Translated and edited by Hubert C. Kennedy

SELECTED WORKS OF GIUSEPPE PEANO

In the decade before 1900, the Italian mathematician Giuseppe Peano was one of the most original and influential pioneers of modern mathematical logic. He made significant contributions to the development of the foundations of mathematics and the axiomatic method (for example, his postulates for the natural numbers), dimension theory (including the space-filling curve), measure theory, vector analysis, differential equations, and the rigorization of analysis.

Several of Peano's works have been translated into other languages; here for the first time is a generous selection of works translated into English. Fifteen articles, one booklet, and parts of two books and one monograph, published between 1883 and 1921, chosen with the interests of mathematicians and logicians in mind, are included. Each selection is preceded by an introductory note. The volume also contains a biographical sketch, a chronological list of Peano's publications (larger by one fifth than any previously published list), and a bibliography on the life and work of Peano. This selection will appeal especially to historians of mathematics and logic, but also to those mathematicians and logicians who wish to know more about how their subject came to be.

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LIFE AND WORKS

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Biographical sketch of Giuseppe Peano

Giuseppe Peano was born on 27 August 1858, the second of the five children of Bartolomeo and Rosa Cavallo. His brother Michele was seven years older. There were to be two younger brothers, Francesco and Bartolomeo, and a sister, Rosa. Peano's first home was the farm 'Tetti Galant,' near the village of Spinetta, three miles from Cuneo and fifty miles south of Turin. When Peano entered school, both he and his brother walked the distance to Cuneo each day, but the family later moved to Cuneo so that the children would not have so far to go. The older brother became a successful accountant and remained in Cuneo. In 1967 Tetti Galant was still in the possession of the Peano family.

Peano's maternal uncle, Michele Cavallo, a priest and lawyer, lived in Turin. On his invitation, Peano moved to that city when he was twelve or thirteen years old. There he received private lessons, some from his uncle, and studied on his own, so that he was able to pass the lower secondary exam of the Cavour School in 1873. He then attended as a regular pupil, completing the upper secondary program in 1876. His performance won him a room-and-board scholarship at the Collegio delle Provincie, which was established to assist students from the provinces to attend the university. Peano duly enrolled at the university in the fall of 1876.

Peano's professors of mathematics at the University of Turin included Enrico D'Ovidio, Angelo Genocchi, Francesco Siacci, Giuseppe Basso, Francesco Faà di Bruno, and Giuseppe Erba. On 16 July 1880 he completed his final exam 'with high honours.' For the academic year 1880–81 he was assistant to Enrico D'Ovidio, and from the fall of 1881 he was assistant to and later substitute for Angelo Genocchi, until Genocchi's death in 1889.

On 21 July 1887 Peano married Carola Crosio, daughter of Luigi Crosio, a genre painter, especially of Pompeian and seventeenth-century scenes. She was interested in opera and they often went to performances at the Royal Theatre of Turin, where the first performances of *Manon*

Lescaut and *Bohème* of Puccini (who was almost exactly Peano's age) were given in 1893 and 1896, respectively.

On 1 December 1890, after regular competition, Peano was named Extraordinary Professor of Infinitesimal Calculus at the University of Turin. He was promoted to Ordinary Professor in 1895. He had already, in 1886, been named Professor at the Military Academy, which was situated close to the university. He gave up his position at the Military Academy in 1901, but retained his position of Professor at the university until his death in 1932, transferring in 1931 to the Chair of Complementary Mathematics. He was elected to membership in a number of scientific societies, among them the Academy of Sciences of Turin, in which he played a very active role. He was also made a knight of the Crown of Italy and of the Order of Saints Maurizio and Lazzaro. Although he was not active politically, his views tended toward socialism, and he once invited a group of textile workers, who were on strike, to a party at his home. During the First World War he advocated a closer federation of the allied countries, to better prosecute the war and, after the peace, to form the nucleus of a world federation. In religion he was Catholic, but non-practising.

Peano's father died in 1888; his mother, in 1910. Although he was rather frail as a child, Peano's health was generally good, his most serious illness being an attack of smallpox in August 1889. Peano died of a heart attack the morning of 20 April 1932, after having taught his regular class the previous afternoon. At his request the funeral was very simple, and he was buried in the Turin General Cemetery. Peano was survived by his wife, his sister, and a brother. He had no children. In 1963 his remains were transferred to the family tomb in Spinetta.

Peano is perhaps most widely known as a pioneer of symbolic logic and a promoter of the axiomatic method, but he considered his work in analysis most important. In 1915 he printed a list of his publications, adding: 'My works refer especially to infinitesimal calculus, and they have not been entirely useless, seeing that, in the judgment of competent persons, they contributed to the constitution of this science as we have it today.' This 'judgment of competent persons' refers in part to the *Encyklopädie der mathematischen Wissenschaften*, in which Pringsheim lists two of Peano's books among the nineteen most important calculus texts since the time of Euler and Cauchy. The first of these was Peano's first major publication and is something of an oddity in the history of mathematics, since the title page gives the author as Angelo Genocchi, not Peano. The title is 'Differential Calculus and Fundamentals of Integral Calculus' and the title page states that it is 'published with additions by Dr Giuseppe Peano.' An explanation of the origin of this book is given in a letter that Peano wrote Genocchi on 7 June 1883:

Esteemed professor,

A few days ago I was at Bocca book-publishers and the director, who is named Lerda, I believe, showed a great desire to publish part of a calculus text during the summer vacation, either written by you or according to your method of teaching: there is no need for me to add how useful such a work would be.

Would you please be so kind as to let me know whether it would be possible to firm up this matter in some way, i.e., if you do not wish to publish the text yourself, perhaps you might think it possible for me to write it, following your lessons, and if I did that, would you be willing to examine my manuscript before publication, or at any rate give me your valued suggestions and look over the proofs of the fascicles as they come off the press; or indeed, in the case that you do not wish to publish this yourself, it might not displease you if I went ahead and published the text, saying that it was *compiled according to your lessons*, or at least citing your name in the Preface, because a great part of the treatment of the material would be yours, inasmuch as I learned it from you.

Please allow me, learned professor, to make a point of saying that I will do my best to see that all comes out well, and believe me

your most devoted student

G. Peano
Turin, 7 June 1883

Genocchi's reception of this letter is expressed in a letter sent to his friend Judge P. Agnelli two months later. In part, he says: 'I could make a complete calculus course with my lessons, but I don't feel up to writing it ... and seeing that Dr Peano, my assistant and substitute, and former student, is willing to take on the trouble, I thought it best not to oppose the project and let him handle it as best he can.'

The publishing firm of Bocca Brothers announced publication for October 1883 of the first volume of Genocchi's *Corso di calcolo infinitesimale* (Course in Infinitesimal Calculus), edited by Peano. When the publication did not appear on time, Felice Casorati wrote to Peano asking about it. A very friendly exchange of letters followed. Peano replied that about 100 pages had been printed and were already being used by his students. The complete book, however, did not appear until the fall of 1884. Before the end of that year Genocchi had had published in mathematical journals of Italy, France, and Belgium the following declaration:

Recently the publishing firm of Bocca Brothers published a volume entitled *Calcolo differenziale e principii di calcolo integrale* (Differential Calculus and Fundamentals of Integral Calculus). At the top of the title page is placed my name, and in the Preface it is stated that besides the course given by me at the University of Turin the volume contains important additions, some modifications, and various annotations, which are placed first. So that nothing will be attributed to me which is not mine, I must declare that I have had no part in the compilation of the aforementioned book and that everything is due to that outstanding young man Dr Giuseppe Peano, whose name is signed to the Preface and Annotations.

Angelo Genocchi

It is easy to imagine that, on reading this declaration, many people thought that Peano was using the name of Genocchi (who was fairly well known in Europe at that time) just to get his own research published, or at least was guilty of bad faith in dealing with his former teacher. Indeed, Charles Hermite, who had written to Genocchi on 6 October 1884 congratulating him on its publication, wrote again on 31 October to commiserate with Genocchi on having such an indiscreet and unfaithful assistant. Genocchi complained to his friend Placido Tardy, who wrote to Genocchi on 8 November 1884: 'After your letter, I sent a visiting card as a thank you note to Peano. I am surprised by what you tell me about the manner in which he has conducted himself toward you, in publishing your lessons.' But it seems that Genocchi, who had a reputation for being quick-tempered, soon recovered and three weeks later Tardy could write: 'I, too, am persuaded that Peano never suspected that he might be lacking in regard toward you.'

In the Preface to his second book, *Applicazioni geometriche del calcolo infinitesimale* (Geometrical Applications of the Infinitesimal Calculus), Peano explicitly states that the first book was authorized by Genocchi, but that he himself assumed full responsibility for it. After Genocchi's death, Peano wrote a necrology of Genocchi for publication in the university yearbook, in which he reaffirms this, and justifies the changes made by the approval the book obtained from critics.

Of the many notable things in this book, the *Encyklopädie der mathematischen Wissenschaften* cites: theorems and remarks on limits of indeterminate expressions, and the pointing out of errors in the better texts then in use; a generalization of the mean-value theorem for derivatives; a theorem on uniform continuity of functions of several variables; theorems on the existence and differentiability of implicit functions; an example of a function whose partial derivatives do not commute; conditions for expressing a function of several variables by a Taylor's formula; a counterexample to the current theory of minima; and rules for integrating rational functions when roots of the denominator are not known.

The other text of Peano cited in the *Encyklopädie* was the two-volume *Lezioni di analisi infinitesimale* (Lessons in Infinitesimal Analysis) of 1893. This work contains fewer new results, but is notable for its rigour and clarity of exposition.

Peano began his scientific publication in 1881–82 with four articles dealing with geometry and algebraic forms. They were along the lines of work done by his professors E. D'Ovidio and F. Faà di Bruno. His work in analysis began in 1883 with 'On the integrability of functions' (5). (The number in parentheses refers to the Chronological List, where the original title of the article is given.) This note contains original notions of integrals and areas. Peano was the first to show that the first-order differential equation $y' = f(x, y)$ is soluble on the sole assumption that f is continuous. His first proof (9) dates from 1886, but its rigour leaves something to be desired. This result was generalized in 1890 to systems of differential equations (27) using a different method of proof. This work is notable, also, for containing the first explicit statement of the axiom of choice. Peano rejected it as being outside the ordinary logic used in mathematical proofs.

Already in the 'Calculus' of 1884 Peano had given many counterexamples to commonly accepted notions in mathematics, but his most famous example was the space-filling curve (24) published in 1890. This curve is given by continuous parametric functions and goes through every point in a square as the parameter ranges over some interval. It can be defined as the limit of a sequence of more 'ordinary' curves. Peano was so proud of this discovery that he had one of the curves in the sequence put on the terrace of his home, in black tiles on white. Some of Peano's work in analysis was quite original and he has not always been given credit for his priority, but much of his publication was designed to clarify definitions and theories then current and make them more rigorous.

Peano's work in logic and the foundations of mathematics may be considered together, although he never subscribed to Bertrand Russell's reduction of mathematics to logic. Peano's first publication in logic was a twenty-page preliminary section, 'Operations of deductive logic,' in his book of 1888, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann* (Geometrical Calculus according to the *Ausdehnungslehre* of H. Grassmann). This section has almost no connection with what follows it. It is a synthesis of, and improvement on, some of the work of Boole, Schröder, Peirce, and MacColl. But by the following year, with the publication of *Arithmetices principia, nova methodo exposita* (The Principles of Arithmetic, Presented by a New Method) (16), Peano has not only improved his logical symbolism, but uses his new method to achieve important new results in mathematics, for this short booklet contains Peano's first statement of his famous postulates for the natural numbers, perhaps the best known of all his creations. His research was done independently of the work of Dedekind, who had the previous year published an analysis of the natural numbers which was essentially that of Peano, without, however, the same clarity. Peano's work also made important innovations in logical notation: ε for set membership, and a new notation for universal quantification. Peano's influence on the development of logical notation was great. Jean van Heijenoort has said: 'The ease with which we read Peano's booklet today shows how much of his notation has found its way, either directly or in a somewhat modified form, into contemporary logic.' (*From Frege to Gödel, a Source Book in Mathematical Logic, 1879–1931*, [Cambridge, Mass., 1967], p. 84. This book contains a translation of part of Peano's 'Principles of Arithmetic.')

Peano was less interested in logic itself than in logic used in mathematics. For this reason he referred to his system as 'mathematical logic.' This may partly explain his failure to develop rules of inference. Some of Peano's omissions are quite remarkable, but his influence on the development of logic was tremendous. Hans Freudenthal said of the Paris Philosophical Congress of 1900: 'In the field of the philosophy of science the Italian phalanx was supreme: Peano, Burali-Forti, Padoa, Pieri absolutely dominated the discussion. For Russell, who read a paper that was philosophical in the worst sense, Paris was the road to Damascus.' ('The Main Trends in the Foundations of Geometry in the 19th Century,' *Logic, Methodology and Philosophy of Science* [Stanford, 1962], p. 616.) Indeed, Russell has remarked that the two men who most influenced his philosophical development were G.E. Moore and Peano.

In 1891 Peano founded the journal *Rivista di matematica*, in which the results of his research, and that of his followers, in logic and the foundations of mathematics were published. It was in this journal, in 1892, that he announced the Formulario project, a project which was to take much of his mathematical and editorial energies for the next sixteen years. The end result of this project would be, he hoped, the publication of a collection of all known theorems in the various branches of mathematics. The notations of his mathematical logic were to be used and proofs of the theorems were to be given. There were five editions of the *Formulario*: the first appeared in 1895, and the last, completed in 1908, contained some 4200 theorems.

In addition to his research in logic and arithmetic, Peano also applied the axiomatic method to other fields, notably geometry, for which he gave several systems of axioms.

Of more importance in geometry was Peano's popularization of the vectorial methods of H. Grassmann, beginning with the publication, already mentioned, in 1888 of the 'Geometrical Calculus according to the *Ausdehnungslehre* of H. Grassmann.' Grassmann's own publications have been criticized for their abstruseness. Nothing could be clearer than Peano's presentation, and the impetus he gave to the Italian school of vector analysis was great.

Enough has been said to show that Peano was interested in, and left his mark on, a number of fields of mathematics. We are making no attempt here to cover everything, but perhaps his work in numerical calculation should be mentioned. Peano devoted a good deal of his effort to this, especially to generalizing Taylor's formula and finding new forms for the remainder. Much of this work has been outmoded by the advent of the high-speed computer, though even in this field something of Peano's influence remains.

The year 1903 saw the beginning of a shift in Peano's interest, from mathematics to the promotion of an international auxiliary language, this shift being nearly completed in 1908 with his election to the presidency of the old Volapük Academy, later renamed 'Academia pro Interlingua.' For the rest of his life Peano was an ardent apostle of the idea of an international auxiliary language. His own invention, *Latino sine flexione* or Latin without grammar, was very similar to that later adopted by the International Auxiliary Language Association and called 'Interlingua.' Peano was an important figure in the history of the artificial language movement. The publication in 1915 of his *Vocabulario commune ad latino-italiano-français-english-deutsch* (Vocabulary Common to Latin-Italian-French-English-German) was a prime event.

It has been said that the apostle in Peano impeded the work of the mathematician. This is no doubt true, especially of his later years, but there can be no question of his very real influence on the development of mathematics. He contributed in no small part to the popularity of the axiomatic method, and his discovery of the space-filling curve must be considered remarkable. Although many of his notions, such as area and integral, were 'in the air,' his originality cannot be denied. He was not an imposing person and his gruff voice with its high degree of lalation (the pronunciation of 'r' so that it sounds like 'l') could hardly have been attractive, but his gentle personality commanded respect, and his keen intellect inspired disciples to the end. His most devoted disciple was one of his last, Ugo Cassina, who shared his interest in mathematics and his zeal for Interlingua. Peano was very generous in promoting the work of young mathematicians. During the First World War, when the Academy of Sciences of Turin limited the number of pages of their Proceedings, Peano, who habitually offered for publication the papers of his school-teacher friends, was most affected. Much of Peano's mathematics is now only of historical interest, but his call for clarity and rigour in mathematics and its teaching remains relevant today – and few have expressed this call more forcefully.

X
A space-filling curve (1890, 1908)*

The following selection contains what is probably, after the postulates for the natural numbers, Peano's best-known discovery. The 'curve which completely fills a planar region' was a spectacular counterexample to the commonly accepted notion that an arc of a curve given by continuous parametric functions could be enclosed in an arbitrarily small region. Indeed, here is a curve given by continuous parametric functions, $x = x(t)$ and $y = y(t)$, such that as t varies throughout the unit interval, the graph of the curve includes every point in the unit square.

We couple this selection with an excerpt from the Formulario *of 1908 in which Peano gives a graphical representation of one 'approach' to such a curve. Hilbert had, after the publication of Peano's original paper, given the first such graphical representation, but Peano was probably led to his discovery by just such a representation. He published his result without diagrams because, it would seem, he wanted no one to think that a false proof lurked in a forced interpretation of a diagram. His proof is purely analytic.*

Peano's curve is a mapping of the unit interval onto its Cartesian product. In the development of topology, this gave rise to the study of Peano spaces. *(A Peano space is a Hausdorff space which is an image of the unit interval under a continuous mapping.) It also raised the question: Which spaces can be mapped continuously onto their Cartesian product? It is of interest that Peano's example is unique, in the sense that: 'The only non-degenerate ordered continuum C which admits a mapping $f : C \to C \times C$ onto its square $C^2 = C \times C$ is the real line segment I.'* (S. Mardešić, 'Mapping Ordered Continua onto Product Spaces,' *Glasnik mat.-fiz. astr. drustvo mat. fiz. hrvatske*, (2) 15 (1960), 85–9; p. 88, Theorem 4.)

1 ON A CURVE WHICH COMPLETELY FILLS A PLANAR REGION

In this note we determine two single-valued and continuous functions

* 'Sur une courbe, qui remplit toute une aire plane,' *Mathematische Annalen*, 36 (1890), 157–60 [24], and excerpt from *Formulario mathematico*, vol. 5 (Turin: Bocca, 1905–8), pp. 239, 240 [138].

x and y of a (real) variable t which, as t varies throughout the interval (0, 1), take on all pairs of values such that $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. If, according to common usage, we consider the locus of points whose coordinates are continuous functions of a variable to be a continuous curve, then we have an arc of a curve which goes through every point of a square. Thus, being given an arc of a continuous curve, with no other hypothesis, it is not always possible to enclose it in an arbitrarily small region.

We shall use the number 3 as a base of numeration and refer to each of the numerals 0, 1, 2 as a digit. We now consider the infinite sequence of digits $a_1, a_2, a_3, \ldots$, which we write

$$T = 0 \,.\, a_1 a_2 a_3 \ldots$$

(for the moment, T is merely a sequence of digits).

If a is a digit, we designate by $\mathbf{k}a$ the digit $2 - a$, the *complement* of a; i.e., we let

$$\mathbf{k}0 = 2, \qquad \mathbf{k}1 = 1, \qquad \mathbf{k}2 = 0.$$

If $b = \mathbf{k}a$, we deduce that $a = \mathbf{k}b$. We also have $\mathbf{k}a \equiv a \pmod 2$. We designate by $\mathbf{k}^n a$ the result of the operation $\mathbf{k}$ repeated n times on a. If n is even, we have $\mathbf{k}^n a = a$; if n is odd, $\mathbf{k}^n a = \mathbf{k}a$. If $m \equiv n \pmod 2$, we have $\mathbf{k}^m a = \mathbf{k}^n a$.

We let correspond to the sequence T the two sequences

$$X = 0 \,.\, b_1 b_2 b_3 \ldots, \qquad Y = 0 \,.\, c_1 c_2 c_3 \ldots,$$

where the digits b and c are given by the relations

$$\begin{aligned} &b_1 = a_1, \quad c_1 = \mathbf{k}^{a_1} a_2, \quad b_2 = \mathbf{k}^{a_2} a_3, \quad c_2 = \mathbf{k}^{a_1 + a_3} a_4, \\ &b_3 = \mathbf{k}^{a_2 + a_4} a_5, \ldots, \\ &b_n = \mathbf{k}^{a_2 + a_4 + \ldots + a_{2n-2}} a_{2n-1}, \quad c_n = \mathbf{k}^{a_1 + a_3 + \ldots + a_{2n-1}} a_{2n}. \end{aligned}$$

Thus, b_n, the nth digit of X, is equal to a_{2n-1}, the nth digit of uneven rank in T, or to its complement, according as the sum $a_2 + \ldots + a_{2n-2}$ of digits of even rank, which precede it, is even or odd, and analogously for Y. We may thus write these relations in the form:

$$\begin{aligned} &a_1 = b_1, \quad a_2 = \mathbf{k}^{b_1} c_1, \quad a_3 = \mathbf{k}^{c_1} b_2, \quad a_4 = \mathbf{k}^{b_1 + b_2} c_2, \ldots, \\ &a_{2n-1} = \mathbf{k}^{c_1 + c_2 + \ldots + c_{n-1}} b_n, \quad a_{2n} = \mathbf{k}^{b_1 + b_2 + \ldots + b_n} c_n. \end{aligned}$$

If the sequence T is given, then X and Y are determined, and if X and Y are given, then T is determined.

We give the name *value* of the sequence T to the quantity (analogous to a decimal number having the same notation)

$$t = \text{val } T = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{a_n}{3^n} + \dots$$

To each sequence T corresponds a number t such that $0 \leqslant t \leqslant 1$. For the converse, the numbers t in the interval (0, 1) divide into two classes:

(α) The numbers, different from 0 and 1, which give an integer when multiplied by a power of 3. They are represented by two sequences, the one

$$T = 0 \,.\, a_1 a_2 \dots a_{n-1} a_n 222 \dots,$$

where a_n is equal to 0 or 1, and the other

$$T' = 0 \,.\, a_1 a_2 \dots a_{n-1} a_n' 000 \dots,$$

where $a_n' = a_n + 1$.

(β) The other numbers. These are represented by only one sequence T.

Now, the correspondence established between T and (X, Y) is such that if T and T' are two sequences of different form, but val T = val T', and if X, Y are the sequences corresponding to T, and X', Y' those corresponding to T', we have

$$\text{val } X = \text{val } X', \qquad \text{val } Y = \text{val } Y'.$$

Indeed, consider the sequence

$$T = 0 \,.\, a_1 a_2 \dots a_{2n-3} a_{2n-2} a_{2n-1} a_{2n} 2222 \dots,$$

where a_{2n-1} and a_{2n} are not both equal to 2. This sequence can represent every number of class α. Letting

$$X = 0 \,.\, b_1 b_2 \dots b_{n-1} b_n b_{n+1} \dots$$

we have

$$b_n = \mathbf{k}^{a_2 + \dots + a_{2n-2}} a_{2n-1}, \qquad b_{n+1} = b_{n+2} = \dots = \mathbf{k}^{a_2 + \dots + a_{2n-2} + a_{2n}} 2.$$

Letting T' be the other sequence whose value coincides with val T, we have

$$T' = 0 \,.\, a_1 a_2 \dots a_{2n-3} a_{2n-2} a'_{2n-1} a'_{2n} 0000 \dots$$

and

$$X' = 0 \,.\, b_1 \dots b_{n-1} b'_n b'_{n+1} \dots$$

The first $2n - 2$ digits of T' coincide with those of T; hence the first $n - 1$ digits of X' coincide also with those of X. The others are determined by the relations

$$b_n' = \mathbf{k}^{a_2 + \ldots + a_{2n-2}} a'_{2n-1},$$
$$b'_{n+1} = b'_{n+2} = \ldots = \mathbf{k}^{a_2 + \ldots + a_{2n-2} + a'_{2n}} 0.$$

We now distinguish two cases, according as $a_{2n} < 2$ or $a_{2n} = 2$.

If a_{2n} has the value 0 or 1, we have

$$a'_{2n} = a_{2n} + 1, \qquad a'_{2n-1} = a_{2n-1}, \qquad b_n' = b_n,$$
$$a_2 + a_4 + \ldots + a_{2n-2} + a'_{2n} = a_2 + \ldots + a_{2n-2} + a_{2n} + 1,$$

whence

$$b'_{n+1} = b'_{n+2} = \ldots = b_{n+1} = b_{n+2} = \ldots = \mathbf{k}^{a_2 + \ldots + a_{2n}} 2.$$

In this case the two series X and X' coincide in form and value.

If $a_{2n} = 2$, we have $a_{2n-1} = 0$ or 1, $a'_{2n} = 0$, $a'_{2n-1} = a_{2n-1} + 1$, and on setting

$$s = a_2 + a_4 + \ldots + a_{2n-2}$$

we have

$$b_n = \mathbf{k}^s a_{2n-1}, \qquad b_{n+1} = b_{n+2} = \ldots = \mathbf{k}^s 2,$$
$$b_n' = \mathbf{k}^s a'_{2n-1}, \qquad b'_{n+1} = b'_{n+2} = \ldots = \mathbf{k}^s 0.$$

Now, since $a'_{2n-1} = a_{2n-1} + 1$, the two fractions $0 . a_{2n-1}222\ldots$ and $0 . a'_{2n-1}000\ldots$ have the same value. Operating on the digits with the same operation $\mathbf{k}^s$ we obtain the two fractions

$$0 . b_n b_{n+1} b_{n+2} \ldots \qquad \text{and} \qquad 0 . b_n' b'_{n+1} b'_{n+2} \ldots,$$

which also have the same value, as may easily be seen. Hence, the fractions X and X', although differing in form, have the same value.

Analogously we may show that val $Y' =$ val Y.

Therefore, if we set $x =$ val X, and $y =$ val Y, we deduce that x and y are two single-valued functions of the variable t in the interval $(0, 1)$. Indeed, if t tends to t_0, the first $2n$ digits of the development of t finally coincide with those of the development of t_0 if t_0 is a β, or with those of one of the two developments of t_0 if t_0 is an α; and so the first n digits of the x and y corresponding to t will coincide with those of the x and y corresponding to t_0.

Finally, to each pair (x, y) such that $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$ corresponds at least one pair of sequences (X, Y) which express that value; to (X, Y)

corresponds a T, and to this a t. Thus one may always determine t in such a fashion that the two functions x and y take on any arbitrarily given values in the interval (0, 1).

We arrive at the same results if, in place of 3, we take any odd number whatever as our base of numeration. One could also take an even number as base, but then the correspondence between T and (X, Y) has to be less simple.

We may form an arc of a continuous curve which completely fills a cube. We make correspond to the fraction (to base 3)

$$T = 0 \,.\, a_1a_2a_3a_4 \ldots$$

the fractions

$$X = 0 \,.\, b_1b_2 \ldots, \qquad Y = 0 \,.\, c_1c_2 \ldots, \qquad Z = 0 \,.\, d_1d_2 \ldots,$$

where

$$\begin{aligned} b_1 &= a_1, \qquad c_1 = \mathbf{k}^{b_1}a_2, \qquad d_1 = \mathbf{k}^{b_1+c_1}a_3, \qquad b_2 = \mathbf{k}^{c_1+d_1}a_4, \ldots, \\ b_n &= \mathbf{k}^{c_1+\ldots+c_{n-1}+d_1+\ldots+d_{n-1}}a_{3n-2}, \\ c_n &= \mathbf{k}^{d_1+\ldots+d_{n-1}+b_1+\ldots+b_n}a_{3n-1}, \\ d_n &= \mathbf{k}^{b_1+\ldots+b_n+c_1+\ldots+c_n}a_{3n}. \end{aligned}$$

We then prove that $x = \text{val } X$, $y = \text{val } Y$, $z = \text{val } Z$ are single-valued and continuous functions of the variable $t = \text{val } T$; and that if t varies between 0 and 1, then x, y, z take on all triplets of values which satisfy the conditions $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1$.

Mr Cantor (*Journal de Crelle*, 84 [1878], 242) has shown that a one-to-one correspondence (*unter gegenseitiger Eindeutigkeit*) between the points of a line and those of a surface can be established. But Mr Netto (*J. reine angew. Math.* 86 [1879], 263–8) and others have shown that such a correspondence is necessarily discontinuous. (See also G. Loria, 'La definizione dello spazio ad n dimensioni ... secondo le ricerche di G. Cantor,' *Giornale di matematiche*, 1877.) My note shows that one can establish single-valuedness and continuity from one side, i.e. to the points of a line can be made to correspond the points of a surface in such a fashion that the image of the line is the entire surface, and that the point on the surface is a continuous function of the point on the line. But this correspondence is not one-to-one, for to the points (x, y) of the square, if x and y are βs, then indeed there is only one corresponding t, but if x or y, or both, are αs, then the corresponding values of t are two or four in number.

It has been shown that an arc of a continuous planar curve may be enclosed in an arbitrarily small region:

1. If one of the functions, x for example, coincides with the independent variable t, in which case we have the theorem on the integrability of continuous functions.

2. If the two functions x and y are of limited variation (Jordan, *Cours d'analyse*, III [1887], p. 599). This is not true, however, as the preceding example shows, on the sole supposition of continuity of the functions x and y.

These x and y, the continuous functions of the variable t, are nowhere differentiable.

2 REMARKS ON A SPACE-FILLING CURVE

* 3. $n \varepsilon \mathrm{N}_1 . \supset . \exists\,(\mathrm{Cx}n\mathrm{fq})\,\mathrm{cont} \cap f \ni (f`\mathrm{q} = \mathrm{Cx}n)$

There exists a complex of order n, or a point in n-dimensional space, which is a continuous function of a real variable, or of time, such that the trajectory of the moving point fills the whole space. That is, there exists a continuous curve which goes through every point of a plane, and there exists a curve which goes through every point of a space, etc. This result is of interest in the study of the foundations of geometry, for there does not exist a specific character which distinguishes a curve from a surface.

If we wish, as the variable t varies from 0 to 1, to have the point with coordinates x and y, functions of t, describe the whole square $(\Theta`\Theta)$, we develop t as a decimal fraction, or to some base analogous to the decimal:

$$t = 0\,.\,a_1 a_2 a_3 \ldots$$

where $a_1, a_2, a_3, \ldots$ are digits. If with the digits in the even places we form the number x, and with the digits in the uneven places we form the number y, we have a reciprocal correspondence between one decimal fraction and two other decimal fractions. But two decimal fractions of different form, such as 0.0999 ... and 0.1000 ..., may have the same value; and the correspondence between the number t and the numbers x and y is not continuous. If we decompose the square of side 1 into 100 squares of side 1/10, then if t goes from the values 0.0900 ... $\vdash$ 0.0999 ... to the values 0.1000 ... $\vdash$ 0.1999 ..., the point (x, y) goes from the last square in the first column to the first square in the second column, and these two squares are not adjacent.

We place the partial squares so that they will be adjacent. In base 2 enumeration, we suppose four partial squares to be in the order given in figure (a), and in base 3 in the order of figure (b).

Now we divide every partial square into other squares, and so on *ad* *infinitum.* Figure (c) represents the succession of 16 squares in base 2; figure (d) the succession of 81 squares in base 3.

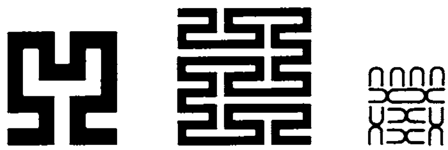
If we represent by the sign $\cap$ the succession $\begin{smallmatrix}1 & 2\\ 0 & 3\end{smallmatrix}$, or figure (a), then figure (e) represents the succession of 64 squares in base 2.

1	2
0	3

(a)

2	3	8
1	4	7
0	5	6

(b)



(c) (d) (e)

In my article 'Sur une courbe qui remplit toute une aire plane,' I gave the analytic expression for the continuous correspondence between the real number t and the complex number $(x; y)$.

See also Hilbert, *Math. Ann.*, 38 (1891), 459; Cesaro, *Bull. sci. math.*, 21 (1897), 257; Moore, *Trans. Amer. Math. Soc.* (1900), 72; Lebesgue, *Leçons sur l'intégration* (Paris, 1904), p. 45.